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Spectral synthesis on torsion groups ☆

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Abstract

Spectral synthesis on Abelian torsion groups is proved.

Keywords: Spectral analysis; Spectral synthesis; Torsion groups

In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is an Abelian group then $\mathbb{C}^G$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the product topology. The dual of $\mathbb{C}^G$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G. This space is also identified with the set of all finitely supported complex valued functions on G in the following obvious way. If the point mass concentrated at the element g is denoted by δ_g, then each measure x in $\mathcal{M}_c(G)$ has a unique representation in the form

$$x = \sum_{g \in G} x(g)\delta_g$$

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with some finitely supported function $x : G \to \mathbb{C}$. "Identification" means that we use the same letter x for both the measure and the representing function. In this sense δ_g is the characteristic function of the singleton $\{g\}$. The pairing between $\mathbb{C}^G$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle x, f\rangle = \sum_{g\in G} x(g)\overline{f(g)}.$$

Convolution on $\mathcal{M}_c(G)$ is defined in the usual way by putting

$$x * y(g) = \sum_{h\in G} x(h)y(gh^{-1})$$

for any x, y in $\mathcal{M}_c(G)$ and g in G. This convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative algebra with unit δ_1. One realizes immediately that the algebra $\mathcal{M}_c(G)$ is identical with the finite group algebra of G. Hence we can use the alternative notation $\mathbb{C}G$ for $\mathcal{M}_c(G)$, which may be more familiar for algebraists. In this sense we can consider G as a subset of $\mathbb{C}G$ by identifying the group element g with the corresponding measure δ_g.

Homomorphisms of G into the additive group of complex numbers, or into the multiplicative group of nonzero complex numbers are called *additive*, or *exponential functions*, respectively. Bounded exponential functions are exactly the *characters* of G. A function of the form $g \mapsto P(a_1(g), a_2(g), \ldots, a_n(g))$ on G is called a *polynomial*, if P is a complex polynomial in n variables and a_k is additive for $k = 1, 2, \ldots, n$. A complex valued function on G is called an *exponential monomial* if it is the product of a polynomial and an exponential. Linear combinations of exponential monomials are called *exponential polynomials*. In particular, linear combinations of characters are called *trigonometric polynomials*.

Translation with the element h in G is the operator mapping any function f in $\mathbb{C}^G$ onto its *translate* $\tau_h f$ defined by $\tau_h f(g) = f(gh)$ for any g in G. A subset of $\mathbb{C}^G$ is called *translation invariant* if it contains all translates of its elements. A closed linear subspace of $\mathbb{C}^G$ is called a *variety* on G if it is translation invariant.

For any variety V on G the *annihilator of* V is the set $V^\perp$ of all measures in $\mathbb{C}G$ which vanish on V. Clearly this is an ideal, which is proper if and only if V is nonzero. Similarly, for any ideal I in $\mathbb{C}G$ the *annihilator of* I is the set $I^\perp$ of all functions in $\mathbb{C}^G$, which are annihilated by all measures in I. Clearly this is a variety on G, which is nonzero if and only if I is proper. Moreover, by the Hahn–Banach theorem it is clear that $V^{\perp\perp} = V$ and $I^{\perp\perp} = I$ holds for each variety on G and for any ideal I in $\mathbb{C}G$. For more details see, e.g., [7].

The basic question of spectral analysis on G can be formulated as follows: does any nonzero variety on G contain an exponential function? If so, then we say that *spectral analysis holds on* G. In [8] the second author proved that if G is an Abelian torsion group then the answer is "yes." Recently in [4] M. Laczkovich and G. Székelyhidi have presented a complete characterization of Abelian groups having spectral analysis: it is necessary and sufficient that the torsion free rank of the group is less than the continuum.

Another basic problem concerns spectral synthesis on G: given a nonzero variety on G, do the exponential monomials in this variety span a dense subspace? If so, then we say that *spectral synthesis holds on* G. This is the case, for instance, if G is a finitely generated free Abelian group by a result of M. Lefranc [5]. In [2] R.J. Elliot presented a theorem stating that spectral synthesis holds for any Abelian group, however his proof was defective. In fact, recently in [9] the second author proved that spectral synthesis fails to hold on any Abelian group with infinite torsion free rank. Based on Lefranc's result one may think that spectral synthesis holds for an Abelian group exactly if it is finitely generated. In fact, in this paper we show that this is not the case: spectral synthesis holds on any Abelian torsion group.

We recall that a *torsion group* is a group in which every element is of finite order. In other words, for every g in the group there exists a positive integer n with $g^n = 1$, the identity of G. This means that among the commutative groups, torsion groups are exactly the groups in which every finite subset generates a finite subgroup. It is easy to see, that on Abelian torsion groups every exponential is a character and every additive function is identically zero, hence on Abelian torsion groups every nonzero exponential monomial is a character and every exponential polynomial is a trigonometric polynomial (see [8]). Consequently, spectral analysis on Abelian torsion groups means that every nonzero variety contains a character, and spectral synthesis means that all characters in every nonzero variety generate a dense subspace in the variety. This latter statement is the main result of our present work. A possible reformulation of this statement is a "Nullstellensatz" (see, e.g., [11]) for trigonometric polynomials on the dual of an Abelian torsion group. In fact, this property characterizes Abelian torsion groups. A by-product of our result is the one which says, that the prime spectrum (see, e.g., Chapter V, Section 1.3 of [6]) of the finite group algebra of an Abelian torsion group is homeomorphic with the dual group.

The following theorem is of fundamental importance.

Theorem 1. *Given an Abelian torsion group, then all characters in a nonzero variety generate a dense subspace in this variety if and only if its annihilator ideal in the finite group algebra of the group is the intersection of all maximal ideals including it.*

Proof. First we prove the following statement: if G is an Abelian torsion group, then each maximal ideal M of the finite group algebra $\mathbb{C}G$ has the following form: there exists a character γ_M of G such that the measure x belongs to M if and only if $\langle x, \gamma_M\rangle = 0$. We remark, that the converse statement is obvious: if M is an ideal of this form, then its annihilator is the one dimensional linear subspace generated by γ_M, hence M clearly cannot be included in any proper ideal.

Suppose now that M is a maximal ideal in $\mathbb{C}G$. Then $\mathbb{C}G/M$ is a field, which is an extension of the complex field, as the natural homomorphism of $\mathbb{C}G$ onto $\mathbb{C}G/M$ restricted to the constant multiples of the identity in $\mathbb{C}G$ sets up a field isomorphism onto a subfield of $\mathbb{C}G/M$, which is isomorphic to $\mathbb{C}$. On the other hand, let g be any element of G and n a positive integer with the property that $g^n = 1$. Then

$$\delta_g^{*n} = \delta_{g^n} = \delta_1,$$

and hence

$$\Phi(\delta_g)^n = \Phi\left(\delta_g^{*n}\right) = \Phi(\delta_1) = 1,$$

consequently $\Phi(\delta_g)$ is a complex nth root of unity for any g in G. In particular, $\Phi(\delta_g)$ is a complex number, hence the function $\gamma : G \to \mathbb{C}$ defined by

$$\gamma(g) = \Phi(\delta_g)$$

for each g in G is a homomorphism of G into the complex unit circle, that is, a character of G. Clearly x belongs to M if and only if $\Phi(x) = 0$, which means that $\langle x, \bar{\gamma} \rangle = 0$. Choosing $\gamma_M = \bar{\gamma}$ our first statement is proved.

Suppose now that all characters in the nonzero variety V generate a dense subspace in V and let I denote the annihilator of V, which is a proper ideal in $\mathbb{C}G$. If x is any element belonging to each maximal ideal including I, then the above considerations show that x annihilates all characters which are included in V, and by our assumption, it follows that x annihilates V. Hence x belongs to I. Conversely, suppose that I, the annihilator of V is the intersection of all maximal ideals including I. Suppose that the subvariety generated by all characters in V is smaller than V. Then by the Hahn–Banach theorem there exists a linear functional x in $\mathbb{C}G$ which annihilates all characters in V, but it does not belong to I. Annihilating all characters in V means that x belongs to all maximal ideals including I, hence, by our assumption it must belong to I, which is a contradiction and our theorem is proved.

We shall make use of the following result. It follows from some known results on discrete spectral synthesis (see, e.g., [10]), but here we give an independent proof.

Theorem 2. *Spectral synthesis holds on any finite Abelian group.*

Proof. Let G be a finite Abelian group, then the characters $\gamma_1, \gamma_2, \ldots, \gamma_n$ of G form a basis for $\mathbb{C}^G$. Let V be any proper variety in $\mathbb{C}^G$, then any function f in V has a unique representation of the form

$$f(x) = \sum_{i=1}^{n} c_i \gamma_i(x).$$

Obviously it is enough to show that here $c_i\gamma_i$ belongs to V for $i = 1, 2, \ldots, n$. Denote the group elements by $y_1, y_2, \ldots, y_n$. As the characters are linearly independent, the matrix $(\gamma_i(y_j))$ is nonsingular, hence from the system of linear equations

$$f(x + y_j) = \sum_{i=1}^{n} \gamma_i(y_j) c_i \gamma_i(x),$$

which holds for each x in G and for $j = 1, 2, \ldots, n$, each term $c_i\gamma_i$ can be expressed as a linear combination of translates of f. This means that if $c_i \neq 0$ then γ_i is in V and the statement is proved. □

We remark that this theorem is also a simple consequence of the following facts. If G is a finite Abelian group, then $\mathbb{C}G$ is a semi-simple Artin ring and so is any homomorphic image of it. Hence, if I is any ideal in $\mathbb{C}G$, then the intersection of all maximal ideals, that is, the Jacobson radical in $\mathbb{C}G/I$ is zero, therefore the intersection of all maximal ideals including I is equal to I.

Our main result follows.

Theorem 3. *Spectral synthesis holds on any Abelian torsion group.*

Proof. Let V be a proper variety in $\mathbb{C}^G$ and let W denote the linear span of the set of all characters contained in V. We have to prove that W is dense in V. Supposing the contrary there exists a measure x in $\mathcal{M}_c(G)$ such that $\langle x, \gamma\rangle = 0$ whenever γ is a character in V, but $\langle x, f_0\rangle \neq 0$ for some f_0 in V.

Let J denote the support of x; then J is a finite subset of G. Let $\mathcal{H}$ denote the family of all finite subgroups of G containing J. For every H in $\mathcal{H}$ let V_H denote the set of the restrictions of the elements of V to H. It is easy to check that V_H is a variety in $\mathbb{C}^H$. Whenever a function Φ is defined on J then we put $\langle x, \Phi\rangle = \sum_{g\in J} x(g)\Phi(g)$. If H is in $\mathcal{H}$ then $\langle x, f_0|H\rangle = \langle x, f_0\rangle \neq 0$. Since spectral synthesis holds in H and $f_0|H$ belongs to V_H, there is a character γ_H of H such that γ_H belongs to V_H and $\langle x, \gamma_H\rangle \neq 0$.

Hence we have a net (γ_H) along the directed set $\mathcal{H}$ in the product space $\mathbb{T}^G$. From its compactness it follows that it has an accumulation point, that is, there is a function $\gamma_0 : G \to \mathbb{T}$ such that for every finite subset F of G and for every $\epsilon > 0$ there exists an H in $\mathcal{H}$ with $F \cup J$ is included in H and $|\gamma_0(g) - \gamma_H(g)| < \epsilon$ holds for each g in F. It is clear that γ_0 is a character of G. As V is closed, we also have that γ_0 belongs to V.

Since each element g in J has a finite order, the set of values $\gamma(g)$, where γ is a character and g is in J is finite. This implies that the set $\langle x, \gamma_H\rangle$ for H in $\mathcal{H}$ is a finite set of complex numbers. As $\langle x, \gamma_0\rangle$ is one of these numbers it follows $\langle x, \gamma_0\rangle \neq 0$. This, however, contradicts the fact that γ_0 is in V. □

Alternatively, one could use some known ring-theoretic results to show that for an Abelian torsion group G, every ideal in the group algebra $\mathbb{C}G$ is an intersection of maximal ideals. First of all, it is a simple observation that for such a group G, every prime ideal of $\mathbb{C}G$ is a maximal ideal. For if P is a prime ideal in $\mathbb{C}G$ and g is an element of G then $0 = g^k - 1 = (g-1)(g-\eta)\dots(g-\eta^{k-1})$ belongs to P with some positive integer k and a complex primitive kth root of unity η, hence some $g - \eta^m$ is in P. In particular g is in $P + \mathbb{C}$ and this is true for all the elements of G, so $\mathbb{C}G/P = (P+\mathbb{C})/P \simeq \mathbb{C}$. And having a field as quotient ring, P has to be maximal.

Now since all prime ideals are maximal, our statement reduces to the property that every ideal in the group algebra $\mathbb{C}G$ is an intersection of prime ideals, that is, every ideal is semi-prime. However, this property is a known characterization of the so-called fully idempotent rings, which are by definition the rings where $I^2 = I$ for all ideals I. (For more details and a proof of this characterization, see, e.g., [12].) In particular, a regular ring is always fully idempotent (as if x is in the ideal I of the ring R and it satisfies $x = xyx$ with some ring element y then clearly x belongs to I^2), and by a theorem of Auslander (see [1]) a commutative complex group algebra $\mathbb{C}G$ is regular if and only if G is torsion.

Conversely, a commutative group G is necessarily torsion if the group algebra $\mathbb{C}G$ is fully idempotent. This also follows from Auslander's theorem, with the observation that commutative and fully idempotent rings are regular. (For $x \in R$, $x \in xR = (xR)^2 = x^2R$, hence $x = x^2y$ with some $y \in R$.)

Thus the property, that each proper ideal of the commutative group algebra is the intersection of all maximal ideals including it characterizes Abelian torsion groups as the following theorem states.

Theorem 4. *Let G be an Abelian group. Then G is a torsion group if and only if each proper ideal of the complex group algebra $\mathbb{C}G$ is an intersection of maximal ideals.*

We can reformulate Theorem 3 in the following way, obtaining an analogue of Hilbert's Nullstellensatz (see [11]).

Theorem 5 ("Nullstellensatz"). *Suppose, that a nonempty set S of trigonometric polynomials on a 0-dimensional compact Abelian group is given, and another trigonometric polynomial p is zero on all the common roots of the ones belonging to S. Then p is contained in the ideal generated by the set S.*

Proof. By duality theory any 0-dimensional compact Abelian group is the dual of an Abelian torsion group G (see, e.g., [3]). Any trigonometric polynomial on $\hat{G}$ is a finite linear combination of characters of $\hat{G}$, that is, the Fourier transform of a finitely supported measure on G. Hence the statement of the present theorem can be reformulated in the following way: if a finitely supported measure on G annihilates all characters, which are annihilated by a given nonempty set of finitely supported measures, then it belongs to the ideal in $\mathbb{C}G$ generated by the given set. But this is exactly spectral synthesis on G and our theorem is proved. □

Another consequence of the above considerations is summarized in the following theorem.

Theorem 6. *For any Abelian torsion group the prime spectrum of the finite group algebra is homeomorphic with the dual group.*

Proof. We have to prove that if G is an Abelian torsion group, then the Zariski topology (see [6]) on the prime spectrum of $\mathbb{C}G$ is identical with the Pontryagin topology on the character group (see [3]).

Earlier we already pointed out that if G is an Abelian torsion group then every prime ideal of the group algebra $\mathbb{C}G$ is maximal. In other words, every point in the prime spectrum of $\mathbb{C}G$ is a closed point. Now we will show that any two points in this prime spectrum can be separated by open sets. So let M_1 and M_2 be two different points in the prime spectrum, that is, two different maximal ideals. Denote the corresponding characters by γ_1 and γ_2, respectively, and fix an element g in G with $\gamma_1(g) \neq \gamma_2(g)$. Denote the order of g by k and let η be a primitive kth root of unity in the complex field. As we saw before, there exist nonnegative integers n and m (with both less than k) such that $g - \eta^m$ is in M_1 and $g - \eta^n$ is in M_2. (In fact, $\gamma_1(g) = \eta^m$ and $\gamma_2(g) = \eta^n$.) Moreover, if $\mathcal{M}_r$ denotes in general the set of maximal ideals containing $g - \eta^r$ then $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_{k-1}$ is a partition of the prime spectrum into disjoint sets such that M_1 belongs to $\mathcal{M}_m$ and M_2 belongs to $\mathcal{M}_n$. We claim that the sets $\mathcal{M}_i$ are clopen sets in the Zariski topology. Fix an integer i with $0 \leqslant i \leqslant k-1$ and let I be the intersection of the members of $\mathcal{M}_i$. Then I is also an ideal of the group algebra, and the fact that $g-\eta^i$ is in I guarantees that the set of maximal ideals containing I is exactly $\mathcal{M}_i$. Indeed, if I is contained in a maximal ideal M which belongs to $\mathcal{M}_j$ and $i \neq j$, then $g-\eta^i$ and $g-\eta^j$ are both contained in M, which is not possible. This shows that each set $\mathcal{M}_i$ is closed in the Zariski topology, and since their disjoint union is the whole prime spectrum, they are all open as well. Thus $\mathcal{M}_m$ and $\mathcal{M}_n$ are two disjoint open sets separating M_1 and M_2. This argument shows that the prime spectrum of $\mathbb{C}G$ with the Zariski topology is a Haussdorff topological space.

We have proved that the Zariski topology, which is obviously weaker than the Pontryagin topology, is Haussdorff. We show that actually the two topologies are the same. Indeed, any Pontryagin-closed subset is Pontryagin-compact, as the Pontryagin topology is compact Haussdorff. As the Zariski topology is weaker, this set is also Zariski-compact, because any Zariski-open covering is a Pontryagin-open covering, too. But the Zariski topology is also compact Haussdorff, hence Zariski-compact sets are Zariski-closed, and our theorem is proved. $\square$

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